injection. The mean momentum equation then yields the distribution of shear stress across the inner region, with the parameter a_1 used to calculate the distribution of the turbulent kinetic energy. Finally, the experimentally determined parameter a_2, as indicated in Fig. 2, yields the distribution of the turbulent transport of turbulent kinetic energy across the inner region of the boundary layer.

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Piecewise Uniform Optimum Design for Axial Vibration Requirement

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Introduction

PIECEWISE uniform design of clamped bars for a fundamental frequency requirement has been studied by Turner,[1] Sheu,[2] Sippel and Warner.[3] However, the methods used by these authors do not permit one to find an exact solution for more than two uniform regions in the rod in any reasonably practical way because of the rapid increase in complexity of the general frequency relation. Matrix and steepest gradient methods have been used to get an approximate solution. Instead, the objective of this Note is to show that a closed form of the exact solution exists for an arbitrary number of regions. Then it is used to show the convergence of discrete optimization to the continuous one.

1. Governing Equations

The system (Fig. 1) consists of n segments, length L_i, linear mass density m_i. For axial vibration of fundamental circular

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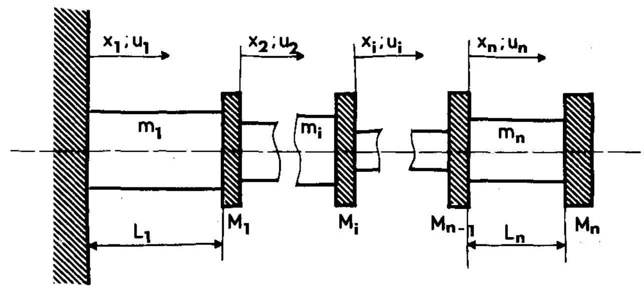

Fig. 1 Clamped bar with n uniform elements.

frequency ω, the mode shape for each member is $u_i(x_i)$, $(0 \leqq x_i \leqq L_i)$, given by the modal equation and boundary conditions

$$u_i''(x_i)+\beta^2 u_i(x_i) = 0 \qquad (i = 1,\ldots,n)$$

$$u_{i-1}(L_{i-1}) = u_i(0) \qquad (i = 2,\ldots,n)$$

$$m_{i-1}u'(L_{i-1})-\beta^2 M_{i-1}u_{i-1}(L_{i-1})-m_i u_i'(0) = 0 \qquad (i = 2,\ldots,n)$$

$$u_1(0) = 0$$

$$m_n u_n'(L_n)-\beta^2 M_n u_n(L_n) = 0$$

where $\beta^2 = \omega^2 \rho/E$, ρ = material density, E = Young's modulus. Solutions can be expressed as

$$u_i = k_i \sin(\beta x_i+\Phi_i)$$

where

$$0 \leqq x_i \leqq L_i \quad \text{and} \quad 0 \leqq \Phi_i < \pi$$

The k_i's can be eliminated, and the boundary conditions become

$$m_{i+1} = [m_i \cot(\beta L_i+\Phi_i)-\beta M_i]\tan\Phi_{i+1} \qquad (i = 1,\ldots,n-1)$$

For the fundamental mode, $u_i > 0$ for $0 < x_i < L_i$ $(i = 1,\ldots,n)$. It is easy to prove that this condition implies

$$0 < \beta L_i+\Phi_i < \pi/2 \qquad (i = 1,\ldots,n)$$

2. Optimization Procedure

We let

$$L = \sum_{i=1}^{n} L_i, \qquad M = \sum_{i=1}^{n} M_i, \qquad \alpha_i = \beta L_i$$

and we nondimensionalize in the following way:

$$l_i = L_i/L, \qquad \Omega_i = \beta M_i L/M, \qquad \mu_i = m_i L/M$$

The optimization problem, which is to minimize the total bar structural mass for the given fundamental frequency ω, can be formulated as follows. Minimize

$$\mathcal{M} = \sum_{i=1}^{n} l_i \mu_i$$

in the $2n$-dimensional (μ_i, Φ_i) space, subject to the constraints

$$\mu_i \cot\Phi_i-\mu_{i-1}\cot(\alpha_{i-1}+\Phi_{i-1})+\Omega_{i-1} = 0 \qquad (i = 2,\ldots,n) \quad (1)$$

$$\Phi_1 = 0 \quad (2)$$

$$-\mu_n \cot(\alpha_n+\Phi_n)+\Omega_n = 0 \quad (3)$$

$$\Phi_i \geqq 0 \quad (4)$$

$$\alpha_i+\Phi_i < \pi/2 \quad (5)$$

$$\mu_i > 0 \quad (6)$$

This problem is equivalent to finding the unconstrained minimum of the new function

$$W = \sum_{i=1}^{n} l_i \mu_i+\lambda_1(\mu_2 \cot\Phi_2-\mu_1\cot\Phi_1+\Omega_1)+ \sum_{i=3}^{n} \lambda_{i-1}[\mu_i\cot\Phi_i-\mu_{i-1}\cot(\alpha_{i-1}+\Phi_{i-1})+\Omega_{i-1}]+ \lambda_n[\Omega_n-\mu_n\cot(\alpha_n+\Phi_n)]$$

where the λ_i's are Lagrange multipliers. The extrema are determined from the following system of equations:

$$\frac{\partial W}{\partial \mu_1} = l_1 - \lambda_1 \cot \alpha_1 = 0 \quad (7)$$

$$\frac{\partial W}{\partial \mu_2} = l_2 + \lambda_1 \cot \Phi_2 - \lambda_2 \cot(\alpha_2 + \Phi_2) = 0 \quad (8)$$

$$\frac{\partial W}{\partial \mu_i} = l_i - \lambda_i \cot(\alpha_i + \Phi_i) + \lambda_{i-1} \cot \phi_i = 0 \quad (i = 3, \ldots, n) \quad (9)$$

$$\frac{\partial W}{\partial \Phi_i} = -\lambda_{i-1}/\sin^2 \Phi_i + \lambda_i/\sin^2(\alpha_i + \Phi_i) = 0 \quad (i = 2, \ldots, n) \quad (10)$$

$$\frac{\partial W}{\partial \lambda_i} = 0 \; (i = 1, \ldots, n) \quad \text{(constraint equations)} \quad (11)$$

Since we have not included the inequality constraints (4–6) in the function W, solutions to Eqs. (7–11) must be checked to satisfy these inequalities. This is easily done in the computation. Indeed, the advantage of this method is that it decouples the μ_i's and the Φ_i's, and allows one to compute sequentially the Φ_i's starting at Φ_2 up to Φ_n, and to compute the optimum μ_i's, starting at μ_n down to μ_1. The algorithm is as follows: 1) Solve Eq. (7) for λ_1. 2) Solve Eqs. (8) and (10) with $i = 2$ for λ_2. 3) Solve Eqs. (9) and (10) with $i = 3, \ldots, n$ for λ_i. 4) Solve Eqs. (11) for $i = n$ to 1 to get the optimum μ_i's: $\mu_n^*, \mu_{n-1}^*, \ldots, \mu_1^*$.

However, Eqs. (9) and (10) are transcendental and they yield several solutions. It can be shown[4] that at each step we have two solutions for Φ_i, one negative and the other positive. According to inequality (4), the latter one only corresponds to the solution.

3. Application: Comparison of continuous and Piecewise Optimization

We have considered a rod clamped at one end, supporting a concentrated mass at the other. The continuous optimum design for a specified fundamental frequency has been explained by the authors already mentioned. One may wonder, however, what is the relationship between that continuous design and piecewise uniform designs in which the rod is divided in n uniform regions of equal lengths.

Computation was carried out, using the algorithm of Sec. 2 for $n = 2, 3, \ldots, 10$. Figure 2 shows the ratio $r = \mathcal{M}_o/\mathcal{M}_n$, where $\mathcal{M}_n$ is the structural mass for an n-steps solution and $\mathcal{M}_o$, the corresponding one for the continuous optimum solution. It can be noticed that convergence of the piecewise uniform to the continuous design is rather good and that little is gained by increasing the numer of steps beyond 2 or 3.

Conclusion

The method described here takes advantage of the simplicity of the equations relative to axial (or torsional) vibration. Thus, it has no claim to generality. It has been applied in Ref. 4, however, to the flexural vibration problem of similar mechanical systems. Because of the higher order equations, a numerical iterative scheme had to be used. The comparison of continuous and piecewise uniform optimization yielded the same qualitative results as in the axial case.

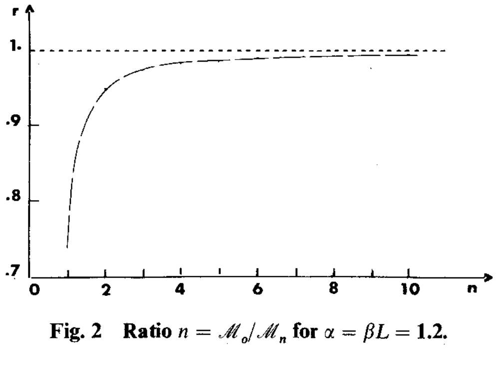

Fig. 2 Ratio $n = \mathcal{M}_o/\mathcal{M}_n$ for $\alpha = \beta L = 1.2$.

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Transient Stresses in Laminated Structures

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Introduction

THE rapid application of surface pressures to a deformable body gives rise to waves which propagate into the interior of the body. The pattern of propagating stress waves in a laminated structure is rather complicated due to multiple reflections and refractions at the interfaces of the laminates, as well as due to multiple reflections at the external boundaries. These repeatedly reflected and refracted stress waves may in fact combine in phase to generate rather high tensile stresses in the interior to cause the delamination failure. Therefore, it is important to perform detail stress analysis on dynamic laminated structure to ascertain that there will be no undesirable high tensile stresses in the interior of the structure.

The analysis of certain classes of the laminated structures can be simplified by representing the mechanical behavior of a laminated medium by a homogeneous continuum model. The simplest homogeneous continuum model is provided by the effective modulus theory.[1,2] For laminated structures subjected to sharp pulse, however, due to the dynamic stresses in a laminated medium is very sensitive to parameters such as pulse shape, the number of layers, the layer thicknesses, the arrangement of laminations, and the material properties, it is necessary to analyze all layers separately in order to arrive at valid results. In this Note numerical results for two multilayered stress-wave attenuators were presented to verify this effect.

Stress-Wave Attenuators

The stress-wave attenuators which are considered here are multilayered slabs with three main elements: a layer of laminated composite, a layer of adhesive, and a back-up plate. The geometries are shown in Fig. 1. The layer of the laminated composite is fabricated from alternating laminates of two different isotropic homogeneous elastic materials. The subscripts 1 and 2 label the field quantities in the layer made of material 1 and in the layer made of material 2, respectively. The field quantities in the adhesive layer and in the back-up plate are labeled by subscripts 3 and 4, respectively. Perfect bond is assumed at the interfaces.

Referring to Fig. 1, the only difference between the design A and the design B is that in the design A the first layer of the

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